Limits of compact decorated graphs*

László Lovász†, Eötvös Loránd University, Budapest
and
Balázs Szegedy, University of Toronto, Toronto

Oct 2010

Contents

Abstract

Following a general program of studying limits of discrete structures, and motivated by the theory of limit objects of converge sequences of dense simple graphs, we study the limit of graph sequences such that every edge is labeled by an element of a compact second-countable Hausdorff space $\mathcal{K}$. The "local structure" of these objects

---

*AMS Subject Classification: Primary 05C99, Secondary 82B99

†Research supported by OTKA grant No. 77780 and ERC Advanced research grant No. 227701

can be explored by a sampling process, which is shown to be equivalent to knowing homomorphism numbers from graphs whose edges are decorated by continuous functions on $\mathcal{K}$. The model includes multigraphs with bounded edge multiplicities, graphs whose edges are weighted with real numbers from a finite interval, edge-colored graphs, and other models. In all these cases, a limit object can be defined in terms of 2-variable functions whose values are probability distributions on $\mathcal{K}$.

1 Introduction

This paper fits into a general program in the frame of which limits of discrete structures are studied. The main assumption is that the objects have a "local structure" which can be explored by certain sampling processes.

A typical example is subsets of integer intervals. If $H \subseteq \{1, 2, 3, \ldots, n\}$ is such an object then we can sample k consecutive elements from $\{1, 2, \ldots, n\}$ uniformly at random and take the intersection of H with it. The corresponding convergence notion leads to limit objects that are shift invariant measures on the compact space $\{0, 1\}^{\mathbb{Z}}$; these measures are important in ergodic theory.

Another example, more relevant for us, is the set of finite simple graphs. For every natural number k there is a sampling process in which we pick k random nodes and look at the subgraph induced by them. A sequence $G_1, G_2, \ldots$ of simple graphs with $|V(G_n)| \to \infty$ is called *convergent* if the distribution of this random induced subgraph is convergent for every k. To every convergent sequence of simple graphs one can assign a limit object in the form of a 2-variable real function [7] (see below).

Our main goal is to generalized these results to limits of multigraph sequences, moments indexed by multigraphs, and beyond. We study the limit of graph sequences such that every edge is labeled by an element of a fixed second-countable compact Hausdorff space $\mathcal{K}$. This includes weighted graphs with bounded edge-weights, or multigraphs with bounded edge multiplicities.

We define convergence of compact decorated graph sequences, and construct limit objects for such sequences. We introduce a notion of homomorphism numbers into such graphs, and show that convergence can be characterized in terms of them.

2 Overview of results

2.1 Decorated graphs

In this section we develop a formalism which provides a unified treatment for many similar problems.

Let S be an arbitrary set. For $n \in \mathbb{Z}_+$, a symmetric map $G: [n]\times[n] \to S$ (i.e., a map satisfying $G(x,y) = G(y,x)$) is called an *S-decorated graph*. We can also think of this as an assignment of an element of S to every edge of $\tilde{K}_n$, the complete graph on the node set $[n] = \{1, 2, \ldots, n\}$ with a loop edge on every node; alternatively, G is an $n \times n$ symmetric matrix with entries from S. We denote by $\mathcal{G}_n(S)$ the set of all S-decorated graphs with n nodes, and set $\mathcal{G}(S) = \cup_{n\geq 0}\mathcal{G}_n(S)$. Often S will have a special element 0, and an edge decorated with 0 will be considered as missing. Often our graphs will have no loops, i.e., $G(x,x) = 0$ for all $x \in [n]$.

If S is finite, then so is $\mathcal{G}_n(S)$. If S is a topological space then, for every n, $\mathcal{G}_n(S)$ is a topological spaces with the product topology. If, in addition, S is compact, then so is $\mathcal{G}_n(S)$.

2.2 Sampling

For every natural number k and $G \in \mathcal{G}(S)$ there is a *sampling process* $\mathbb{G}(G,k)$ which is a random variable whose values are in $\mathcal{G}_k(S)$, and is defined as follows. We pick a random ordered set of k nodes $\{v_1, v_2, \ldots, v_k\}$ uniformly from G and then we create a graph $F = \mathbb{G}(G,k) \in \mathcal{G}_k(S)$ on the node set $\{1, 2, \ldots, k\}$ such that $F_{i,j}$ is G_{v_i,v_j} for every $1 \leq i < j \leq k$.

While $\mathbb{G}(G,k)$ comes with labeled nodes, it is clear that this graph with any other labeling of its nodes arises with the same probability.

2.3 Subgraph densities and moments

From now on, $\mathcal{K}$ denotes a compact separable topological space, and we consider $\mathcal{K}$-decorated graphs. Let $\mathcal{C}$ denote the family of continuous real valued functions on $\mathcal{K}$, and let $\mathcal{F} \subseteq \mathcal{C}$. For an $\mathcal{F}$-decorated graph $F \in \mathcal{G}_k(\mathcal{F})$ and a $\mathcal{K}$-decorated graph $G \in \mathcal{G}(\mathcal{K})$, we introduce the weight $w(f)$ of a function $f: [k] \mapsto V(G)$ by

$$w(f) = \prod_{1\leq i<j\leq k} F_{i,j}(G_{f(i),f(j)}).$$

The *homomorphism number* $\hom(F, G)$ is

$$\hom(F, G) = \sum_{f:[k]\mapsto V(G)} w(f).$$

We also define the *homomorphism density* by

$$t(F, G) = \frac{\hom(F, G)}{|V(G)|^k},$$

which is the expected value $E(w(f))$ for a random map $f:\ [k] \mapsto V(G)$.

2.4 Convergence

Our goal is to study the following convergence notion.

Definition 2.1 An infinite sequence $G_1, G_2, \dots$ of $\mathcal{K}$-decorated graphs is called *convergent* if $|V(G_i)| \to \infty$, and for every k the sampling processes $\{\mathbb{G}(G_i, k)\}_{i=1}^{\infty}$ are weakly convergent in distribution. This means that for every $k \in \mathbb{N}$ and continuous function $f:\ \mathcal{G}_k(\mathcal{K}) \mapsto \mathbb{R}$ the limit $\lim_{i\to\infty} E(f(\mathbb{G}(G_i, k)))$ exists.

We are going to characterize convergence of a graph sequence in terms of homomorphism numbers from $\mathcal{C}$-decorated graphs. Since there are generally too many such graphs, we will also show that we can restrict ourselves to graphs decorated by elements from an appropriate subset of $\mathcal{C}$. We need the following definition.

Definition 2.2 We say that a set $\mathcal{F} \subseteq \mathcal{C}$ is *dense* if for every $\epsilon > 0$ and $f \in \mathcal{C}$ there is an $g \in \mathcal{F}$ such that $|g(x) - f(x)| \le \epsilon$ for every $x \in \mathcal{K}$. We say that $\mathcal{F} \subseteq \mathcal{C}$ is a *generating system* if the linear space generated by the elements of $\mathcal{F}$ is dense.

We will prove the following theorem.

Theorem 2.3 (Equivalence of convergence notions) *Let $(G_1, G_2, \dots)$ be a sequence of $\mathcal{K}$-decorated graphs with $|V(G_n)| \to \infty$, and let $\mathcal{F}$ be a generating system. Then the following are equivalent:*

(i) $(G_1, G_2, \dots)$ *is convergent;*

(ii) *For every $\mathcal{C}$-decorated graph F, the numerical sequence $t(F, G_n)$ is convergent;*

(iii) *For every $\mathcal{F}$-decorated graph F, the numerical sequence $t(F, G_n)$ is convergent.*

2.5 Limit objects

Let $\mathcal{P}(\mathcal{K})$ denote the set of probability Borel measures on the compact space $\mathcal{K}$. The Riesz representation theorem implies that the set $\mathcal{P}(\mathcal{K})$ is a compact topological space with the weak topology. (Recall that the weak topology is the weakest topology such that the function $\mu \mapsto \int_{\mathcal{K}} f \, d\mu$ is continuous for every continuous function $f : \mathcal{K} \to \mathbb{R}$.)

Definition 2.4 We denote by $\mathcal{W}(\mathcal{K})$ the set of two variable Borel measurable functions $W : [0,1]^2 \mapsto \mathcal{P}(\mathcal{K})$ such that $W(x,y) = W(y,x)$ for every $(x,y) \in [0,1]^2$. Elements of $\mathcal{W}(\mathcal{K})$ will be called *$\mathcal{K}$-graphons.*

An important fact about $\mathcal{K}$-graphons is that they can be described by sequences of real-valued measurable functions in the following way. Let W be a $\mathcal{K}$-graphon and let $f \in \mathcal{C}$. Define $W_f : [0,1]^2 \mapsto \mathbb{R}$ by $W_f(x,y) = \int_{\mathcal{K}} f \, dW(x,y)$. The function W_f is a bounded measurable function taking values between $\min(f)$ and $\max(f)$. The Riesz representation theorem implies that the measures $W(x,y)$ are reconstructible from the sequence $\{W_f(x,y)\}_{f \in \mathcal{F}}$ if $\mathcal{F}$ is a generating system.

We will say that $(W_f : f \in \mathcal{F})$ is the *$\mathcal{F}$-moment representation* of W. The name refers to the fact that for various natural choices of $\mathcal{K}$ and $\mathcal{F}$, the numbers $t(F,G)$ $(F \in \mathcal{F})$ behave similarly to the moments of a single-variable function. This analogy is explained and exploited in [9].

Every $\mathcal{K}$-decorated graph G gives rise to a $\mathcal{K}$-graphon W_G as follows. Let $V(G) = [n]$. We split the unit interval into n intervals $J_1, \ldots, J_n$ of length $1/n$, and let $W_G(x,y) = G(i,j)$ for $x \in J_i, y \in J_j$ (here we identify the element $G(i,j) \in \mathcal{K}$ with the distribution concentrated on $G(i,j)$).

For every $\mathcal{K}$-graphon W and $\mathcal{C}$-decorated graph F we introduce the homomorphism density $t(F,W)$ by

$$t(F,W) := \int\limits_{x_1,x_2,\ldots,x_k \in [0,1]} \prod_{1 \le i < j \le k} W_{F_{i,j}}(x_i,x_j) \, dx_1 \, dx_2 \ldots dx_k.$$

It is easy to see that for every $\mathcal{K}$-decorated graph G and $\mathcal{C}$-decorated graph F,

$$t(F,W_G) = t(F,G).$$

Note that if F is $\mathcal{F}$-decorated for some $\mathcal{F} \subseteq \mathcal{C}$, then $t(F,W)$ is expressed in terms of the $\mathcal{F}$-moment representation of W.

We will prove that the limit of a convergent sequence of $\mathcal{K}$-decorated graphs can be represented by a $\mathcal{K}$-graphon:

Theorem 2.5 *Let $\mathcal{F}$ be a countable generating set and let $(G_1, G_2, \dots)$ be a convergent sequence of $\mathcal{K}$-decorated graphs. Then there is a $\mathcal{K}$-graphon W such that $t(F, G_n) \to t(F, W)$ for every $\mathcal{C}$-decorated graph F.*

We in fact prove a more general theorem about convergence of $\mathcal{K}$-graphons.

Theorem 2.6 *Let $\mathcal{F}$ be a countable generating set and let $W_1, W_2, \dots$ be a sequence of $\mathcal{K}$-graphons such that $(t(F, W_n) : \ n = 1, 2, \dots)$ is a convergent sequence for every $F \in \mathcal{G}(\mathcal{F})$. Then there is a $\mathcal{K}$-graphon W such that $t(F, W_n) \to t(F, W)$ for every $F \in \mathcal{G}(\mathcal{C})$.*

2.6 Examples

Example 2.7 (Simple graphs) Let $\mathcal{K}$ be the discrete space with two elements called "edge" and "non edge" or shortly 1 and 0. The set $\mathcal{C}$ consists of all maps $\{0,1\} \to \mathbb{R}$, i.e., of all pairs $(f(0), f(1))$ of real numbers. A natural generating subset (in fact, a basis) in $\mathcal{C}$ consists of the pairs $f_0 = (1,1)$ and $f_1 = (0,1)$. Sampling, convergence, and homomorphism densities correspond to these notions introduced for simple graphs.

Every probability distribution on $\mathcal{K}$ can be represented by a number between 0 and 1 which is the probability of the element "edge". So a $\mathcal{K}$-graphon is described by a symmetric measurable function $W : \ [0,1]^2 \mapsto [0,1]$. This has been the motivating example worked out in [7, 3, 4].

One may, however, take another basis in $\mathcal{C}$, namely the pair $g_0 = (0,1)$ and $g_1 = (1,0)$. Then again $\mathcal{F}$-decorated graphs can be thought of as simple graphs, and $\hom(F, G)$ counts the number of maps that preserve both adjacency and non-adjacency.

Example 2.8 (Multicolored graphs) Let $\mathcal{K}$ be a finite set of "colors" with the discrete topology. Continuous functions on $\mathcal{K}$ can be thought of as vectors in $\mathbb{R}^{\mathcal{K}}$. The standard basis $\mathcal{F}$ in this space corresponds to elements of $\mathcal{K}$, and so $\mathcal{F}$-decorated graphs are just the same as $\mathcal{K}$-decorated graphs. The moment $t(F, G)$ is the probability that a random map $V(F) \to V(G)$ preserves edge colors.

Probability distributions on $\mathcal{K}$ can be described by the probabilities of its points. So a $\mathcal{K}$-graphon is represented by k measurable functions $w_i : \ [0,1]^2 \mapsto [0,1]$ with $\sum_i w_i(x,y) = 1$.

Example 2.9 (Multigraphs) Let G be a multigraph with edge multiplicities at most d. Then G can be thought of as a $\mathcal{K}$-decorated graph, where

$\mathcal{K} = \{0, 1, \ldots, d\}$. This seems equivalent to example 2.8; however, it is more interesting to take a different basis in $\mathbb{R}^{\mathcal{K}}$ in this case, namely the functions $\mathcal{F} = \{1, x, \ldots, x^d\}$. We can represent an $\mathcal{F}$-decorated graph by a multigraph with edge multiplicities at most d, where an edge decorated by x^i is represented by i parallel edges. The advantage of this is that $\hom(F, G)$ is then the number of homomorphisms of F into G as multigraphs.

Example 2.10 (Parallel colored graphs) This example is basically the same as Example 2.8 in the sense that we choose $\mathcal{K}$ to be finite; however we allow an edge to carry more than one color. We take $\mathcal{K} = \{0, 1\}^n$, and let $(e_1, e_2, \ldots, e_n) \in \mathcal{K}$ mean that an edge has color i if and only if $e_i = 1$.

We have a fairly nice basis $\mathcal{F}$: For every vector $x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ we construct a function $f_x : \mathcal{K} \mapsto \{0, 1\}$ such that $f_x(c_1, c_2, \ldots, c_n) = 1$ if and only of $x_i = 1$ implies $c_i = 1$. These functions form a basis in $\mathbb{R}^{\mathcal{K}}$.

Limit objects are more complex. The most natural way is to use $2^n - 1$ measurable functions $w : [0, 1]^2 \mapsto [0, 1]$ whose sum is between 0 and 1.

The main application of this example is that it allows us to study a parallel limit of many graphs on the same node set.

Example 2.11 (Infinitely many parallel graphs) The previous example can be further generalized by allowing infinitely many parallel graphs. Let $\mathcal{K} = \{0, 1\}^{\mathbb{N}}$ be the compact space with the product topology. Everything goes similar to the previous example except that in the definition of $\mathcal{F}$ we only allow finitely many nonzero entries in the vector x to guarantee that the functions $f_x : \mathcal{K} \mapsto \{0, 1\}$ are continuous. Limit objects now can be represented by an $\mathcal{F}$-moment sequence of measurable functions. Every such function is indexed by a finite subset of the natural numbers.

Example 2.12 (Weighted graphs) Let $\mathcal{K} \subseteq \mathbb{R}$ be a bounded closed interval. Let $\mathcal{F}$ be the collection of mononomial functions $x \mapsto x^j$ for $j = 0, 1, 2, \ldots$ on $\mathcal{K}$; then $\mathcal{F}$ is a generating system. It is natural to consider an $\mathcal{F}$-decorated graph F as a multigraph, and then $\hom(F, G)$ is the weighted homomorphism as defined e.g. in [5].

Example 2.13 (Compact topological groups) Let $\mathcal{K}$ be a compact topological group. It is natural to choose $\mathcal{F}$ to be the Pontrjagin dual of $\mathcal{K}$, which is the (discrete) group of continuous homomorphisms from $\mathcal{K}$ to $\mathbb{C}$. In the special case $\mathcal{K} = \mathbb{R}/\mathbb{Z}$, the dual group is isomorphic to the integers, so every $\mathcal{F}$-decorated graph can be considered as graphs with multiple edges such that negative edge multiplicities are allowed.

3 Tools and proofs

3.1 Weak regularity partitions

For a measurable function $W:\ [0,1]^2 \to \mathbb{R}$ we define its *rectangle norm* by

$$\|W\|_\square = \sup_{\substack{A\subseteq[0,1]\\ B\subseteq[0,1]}} \left| \int_A \int_B W(x,y)\,dx\,dy \right| \tag{1}$$

where A and B ranges over all possible measurable subsets of $[0,1]$. It is easy to see that this norm could be defined by the formula

$$\|W\|_\square = \sup_{0\le f,g\le 1} \left| \int_0^1 \int_0^1 W(x,y)f(x)g(y) \right|, \tag{2}$$

where f and g are measurable functions. It is not hard to see that it would not matter much to take the supremum over all functions with bounded absolute value:

$$\|W\|_\square \le \sup_{|f|,|g|\le 1} \left| \int_0^1 \int_0^1 W(x,y)f(x)g(y) \right| \le 4\|W\|_\square. \tag{3}$$

We note that the supremum in the middle is the $L_\infty \to L_1$ operator norm of W as a kernel operator.

The following "weak" version of Szemerédi's Regularity Lemma was proved (in the context of matrices) by Frieze and Kannan [6] (see [8] for this analytic formulation):

Lemma 3.1 *For every $\varepsilon > 0$ there is a constant $k = k(\varepsilon)$ such that for every symmetric measurable function $W \in \mathcal{W}$ there is a partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ of $[0,1]$ into k measurable subsets and a stepfunction W' which is constant on the sets $P_i \times P_j$ such that*

$$\|W - W'\|_\square \le \varepsilon \|W\|_\infty.$$

It is easy to see that (at the cost of increasing $k(\varepsilon, d)$) we can impose additional conditions on the partition $\mathcal{P}$ and the function W'. We can assume that $\mathcal{P}$ refines another partition and that the partition sets have the same measure. We can also assume that $W' = W_\mathcal{P}$, where $W_\mathcal{P}$ is obtained by taking the average of W on each set $P_i \times P_j$. By iterating this we get the following.

Lemma 3.2 *For every $\varepsilon > 0$ and natural numbers t and p there is an integer $k(\varepsilon, t, p) > 0$ such that for every partition $\mathcal{P}$ of $[0, 1]$ into p equal sized measurable subsets and every family of functions $W_i \in \mathcal{W}$ ($i = 1, 2, \ldots, t$), there exists a partition $\mathcal{S} = \{S_1 \ldots, S_k\}$ of $[0, 1]$ into $k = k(\varepsilon, t, p)$ measurable sets such that*

(1) *each S_i has the same measure $1/k$;*

(2) *$\mathcal{S}$ is a refinement of the partition P;*

(3) *$\|W_i - (W_i)_{\mathcal{S}}\|_{\square} \leq \varepsilon \|W_i\|_{\infty}$ holds for each $0 \leq i \leq t$.*

3.2 Equivalence of convergence

We need a few lemmas. For a subset $\mathcal{F} \subseteq \mathcal{C}$ let $\mathcal{F}_n$ denote the set of functions on $\mathcal{K}^n$ whose elements are of the form

$$(c_1, c_2, \ldots, c_n) \mapsto \prod_{i=1}^{n} f_i(c_i)$$

with each f_i in $\mathcal{F}$. It is clear that the elements of $\mathcal{F}_n$ are continuous functions from $\mathcal{K}^n$ to $\mathbb{R}$.

Lemma 3.3 *$\mathcal{C}_n$ is a generating system on $\mathcal{K}^n$.*

Proof. First of all observe that the liner space generated by $\mathcal{C}_n$ is an algebra of continuous functions (containing the constant 1 function) on $\mathcal{K}^n$. Using Stone-Weierstrass Theorem it is enough to show that $\mathcal{C}_n$ is a separating set. Let $c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$ be two distinct elements on $\mathcal{K}^n$ such that $c_i \neq d_i$. Then there is a continuous function f with $f(c_i) \neq f(d_i)$ on $\mathcal{K}$. It is clear that the function $\hat{f}(x_1, x_2, \ldots, x_n) = f(x_i)$ is in $\mathcal{C}_n$ and it separates c form d. $\square$

Lemma 3.4 *If $\mathcal{F}$ is a generating system on $\mathcal{K}$ then so is $\mathcal{F}_n$ on $\mathcal{K}^n$.*

Proof. Let $f : \mathcal{K}^n \mapsto \mathbb{R}$ be an arbitrary continuous function. Then by Lemma 3.3 for every $\epsilon > 0$ there is a finite set of continuous functions $f_{i,j}$ on $\mathcal{K}$ such that

$$\Big|\sum_{i=1}^{k} \prod_{j=1}^{n} f_{i,j}(c_j) - f((c_1, c_2, \ldots, c_n))\Big| \leq \epsilon$$

for every $(c_1, c_2, \ldots, c_n)$ in $\mathcal{K}^n$. Note that in this formula we don't need linear coefficients since they can be "merged" into the terms $f_{i,j}$. Using

that $\mathcal{F}$ is a generating system we have that for every $\epsilon_2 > 0$ there is finite system of functions $\{g_i\}_{i=1}^r$ in $\mathcal{F}$ and real numbers $\lambda_{i,j,k}$ such that

$$|f_{i,j} - \sum_{k=1}^{r} \lambda_{i,j,k} g_k| \leq \epsilon_2$$

everywhere on $\mathcal{K}$ and for every i, j. Now replacing every $f_{i,j}$ by its approximation with the functions g_i we get an approximation of f with linear combinations of elements from $\mathcal{F}_n$ with a precision arbitrary close to ϵ if we let ϵ_2 go to 0. This completes the proof. $\square$

Now we are ready to prove Theorem 2.3. First we observe that the distributional convergence of $\mathbb{G}(G_i, k)$ implies that for any graph F in $\tilde{\mathcal{G}}_k(\mathcal{F})$ the sequence $t(F, G_i)$ is convergent. This follows immediately from the definition of $t(F, G_i)$ since $t(F, G_i) = E(L(\mathbb{G}(G_i, k)))$ for some function L in $\mathcal{F}_{\binom{k}{2}}$ whose components are the edge weights in F.

The other direction follows from Lemma 3.4 since the functions L occurring in the formula above form a generating system.

3.3 Function sequences

An indexed set $s = (s_f : f \in \mathcal{F})$, where $s_f : [0,1]^2 \to \mathbb{R}$ is a bounded symmetric measurable function for each f will be called an *$\mathcal{F}$-indexed function sequence*. Let $\mathcal{S}$ be the set of all $\mathcal{F}$-indexed function sequences. For every $\mathcal{F}$-decorated graph F and $s \in \mathcal{S}$, we can define the "homomorphism density"

$$t(F, s) = \int_{x \in [0,1]^{V(F)}} \prod_{1 \leq i < j \leq k} s_{F_{i,j}}(x_i, x_j)\, dx.$$

The next lemma shows the relation between the $\|.\|_\square$-norm and the homomorphism densities into a sequence of functions.

Lemma 3.5 *Let $u = (u_f : f \in \mathcal{F})$ and $w = (w_f : f \in \mathcal{F})$ be two indexed sets of functions in $\mathcal{W}$, and let $d_f = \max(\|u_f\|_\infty, \|w_f\|_\infty)$. Then for every $\mathcal{F}$-decorated graph F,*

$$|t(F, u) - t(F, w)| \leq 4\Big(\prod_{ij \in E(F)} d_{F_{i,j}}\Big) \sum_{ij \in E(F)} \|u_{F_{i,j}} - w_{F_{i,j}}\|_\square.$$

Proof. Let $E(F) = \{i_1 j_1, \dots, i_m j_m\}$. We have

$$t(F,U) - t(F,W) \\ = \int_{[0,1]^n} \Big(\prod_{r=1}^m W_{F_{i_r,j_r}}(x_{i_r}, x_{j_r}) - \prod_{r=1}^m U_{F_{i_r,j_r}}(x_{i_r}, x_{j_r}) \Big) \, dx.$$

We can write the telescoping sum

$$\prod_{r=1}^m W_{F_{i_r,j_r}}(x_{i_r}, x_{j_r}) - \prod_{r=1}^m U_{F_{i_r,j_r}}(x_{i_r}, x_{j_r}) = \sum_{t=1}^m X_t(x_1, \dots, x_n),$$

where

$$X_t(x_1,\dots,x_n) = \Big(\prod_{r=1}^{t-1} W_{F_{i_r,j_r}}(x_{i_r}, x_{j_r}) \Big) \Big(\prod_{r=t+1}^{m} U_{F_{i_r,j_r}}(x_{i_r}, x_{j_r}) \Big) \\ \times (W_{F_{i_t,j_t}}(x_{i_t}, x_{j_t}) - U_{F_{i_t,j_t}}(x_{i_t}, x_{j_t})).$$

To estimate the integral of a given X_t term, let us integrate first the variables x_{i_t} and x_{j_t}; then by (3),

$$\Big| \int_0^1 \int_0^1 X_t(x_1, \dots, x_n) \, dx_{i_t} \, dx_{j_t} \Big| \le 4 \Big(\prod_{r=1}^m d_{F_{i,j}} \Big) \|U_{F_{i_t,j_t}} - W_{F_{i_t,j_t}}\|_\square,$$

which completes the proof. □

The $\mathcal{F}$-indexed function sequences most important for us will be the $\mathcal{F}$-moment representations of $\mathcal{K}$-graphons. An *$\mathcal{F}$-moment sequence* is a family $(a_f : f \in \mathcal{F})$ of the form

$$a_f = \int_{\mathcal{K}} f \, d\mu$$

where μ is a Borel probability measure on $\mathcal{K}$. An *$\mathcal{F}$-moment function sequence* is a family $(w_f : f \in \mathcal{F})$ of functions $w_f \in \mathcal{W}$ such that $(w_f(x,y) : f \in \mathcal{F})$ is an $\mathcal{F}$-moment sequence for all $x, y \in [0,1]$. Clearly the $\mathcal{F}$-moment representation of a $\mathcal{K}$-graphon is an $\mathcal{F}$-moment function sequence. The next Lemma shows that the converse also holds.

Lemma 3.6 *For every $\mathcal{F}$-moment function sequence w there is a $\mathcal{K}$-graphon W with $\mathcal{F}$-moment representation w.*

Proof. Since $w(x,y)$ is an $\mathcal{F}$-moment sequence for all $x, y \in [0.1]$, there is a probability measure $W(x,y)$ such that $\int_{\mathcal{K}} f \, dW(x,y) = w_f(x,y)$ for

all $f \in \mathcal{F}$. To prove that W is a $\mathcal{K}$-graphon, we have to check that it is measurable as a map $[0,1]^2 \to \mathcal{P}(\mathcal{K})$. Since the Borel sets in $\mathcal{P}(\mathcal{K})$ are generated by the sets $\{\mu : \int_{\mathcal{K}} g\, d\mu \ge 0$ $(g \in \mathcal{C})$, it suffices to check that the sets

$$A_g = \{(x,y) : \int_{\mathcal{K}} g\, dW(x,y) \ge 0$$

are measurable for all $g \in \mathcal{C}$. Since $\mathcal{F}$ is generating, we have functions $g_n \in \mathcal{C}$ such that $\|g_n - g\|_\infty \le 1/n$ and g_n is in the linear hull of $\mathcal{F}$. Clearly $A_g = \cup_n A_{g_n+1/n}$, so it suffices to show that $A_{g_n+1/n}$ is measurable. Let $g_n = \sum_{k=1}^N \alpha_k f_k$, where $f_k \in \mathcal{F}$ and $\alpha_k \in \mathbb{R}$. Then

$$\int_{\mathcal{K}} \Big(g_n + \frac{1}{n}\Big)\, dW(x,y) = \frac{1}{n} + \sum_{k=1}^N \alpha_k \int_{\mathcal{K}} f_k\, dW(x,y) = \frac{1}{n} + \sum_{k=1}^N w_{f_k}(x,y)$$

is a measurable function of (x,y), which proves that $A_{g_n+1/n}$ is measurable. $\square$

3.4 Simultaneous convergence

The following Lemma is our main tool for constructing limit objects.

Lemma 3.7 *Let $\mathcal{F} \subseteq \mathcal{C}$ be a countable generating system. Let Q be a compact convex subset of $\mathbb{R}^{\mathcal{F}}$, and let $s_1, s_2, \dots \in \mathcal{S}$ be a sequence such that $s_n(x,y) \in Q$ for $n = 1,2,\dots$. Assume that there are reals $d_f > 0$ $(f \in \mathcal{F})$ such that $|(s_n)_f(x,y)| \le d_f$ for all n, x and y. Furthermore, assume that $(t(F,s_n) : n = 1,2,\dots)$ is a convergent sequence for every $F \in \mathcal{G}(\mathcal{F})$. Then there is a sequence $w \in \mathcal{S}$ such that $w(x,y) \in Q$ for all x,y and $t(F,s_n) \to t(F,w)$ for every $\mathcal{F}$-decorated graph F.*

Proof. Let $\mathcal{F} = \{f_1, f_2, \dots\}$. For each $t \ge 1$, define $h(t)$ recursively by $h(1) = 1$ and

$$h(t) = k\Big(\frac{1}{t \max\{d_{f_i} : 1 \le i \le t\}}, t, h(t-1)\Big),$$

where k is the function in Lemma 3.2.

For each $t \ge 1$ we construct a partition $\mathcal{P}_t$ of $[0,1]$ into $h(t)$ intervals of equal length, and a subsequence Q_t of the natural numbers by recursion as follows. Let $\mathcal{P}_1 = \{[0,1]\}$ and $Q_1 = \mathbb{N}$. For each n, let $\mathcal{P}_{t,n}$ be a partition refining $\mathcal{P}_{t-1}$ with $h(t)$ partition classes given by Lemma 3.2, when applied

to the sequence $((s_n)_{f_1}, \dots, (s_n)_{f_t})$ and $\varepsilon = 1/t$. We can apply, for each n, a measure preserving transformation to $[0,1]$ so that $\mathcal{P}_{t,n}$ becomes a partition $\mathcal{P}_t$ into intervals. Since $\mathcal{P}_{t,n}$ is a refinement of $\mathcal{P}_{t-1}$, we can do this transformation so that $\mathcal{P}_{t-1}$ remains a partition into intervals.

Let $v_{f,n,t} = ((s_n)_f)_{\mathcal{P}_t}$ and $v_{n,t} = (v_{f,n,t} : f \in \mathcal{F}) \in \mathcal{S}$. We have by Lemma 3.2

$$\|v_{f,n,t} - (s_n)_f\|_\square \le \frac{1}{t}\|(s_n)_f\|_\infty \le \frac{1}{t}d_f. \tag{4}$$

Let Q_t be an infinite subsequence of Q_{t-1} such that for every $1 \le a, b \le h(t)$ and $f \in \mathcal{F}_t$, the functions $v_{f,n,t}$ converge to a function $u_{f,t}$ for $n \to \infty$, $n \in Q_t$. Such a subsequence can be selected since the functions $v_{f,n,t}$ are stepfunctions with a fixed partition and they are uniformly bounded. Let $u_t = (u_{f,t} : f \in \mathcal{F}) \in \mathcal{S}$.

We claim that the functions $u_{f,t}$ converge to some symmetric measurable function w_f almost everywhere if $t \to \infty$. This follows from the properties that $(u_{f,t})_{\mathcal{P}_t} = u_{f,t-1}$ and $|u_{f,t}| \le d_f$. Using the convergence theorem of bounded martingales, one gets the convergence as in [7].

Let $w = (w_f : f \in \mathcal{F})$. It is clear that $w(x,y) \in Q$ for almost all $x, y \in [0,1]$, and we may change the limit functions on a set of measure 0 so that this holds everywhere. For any $\mathcal{F}$-decorated graph F with n nodes, there is a real number A_F such that

$$t(F, s_n) \to A_F \qquad (n \to \infty). \tag{5}$$

We also have, trivially,

$$t(F, u_t) \to t(F, w) \qquad (t \to \infty), \tag{6}$$

and

$$t(F, v_{n,t}) \to t(F, u_t) \qquad (n \to \infty). \tag{7}$$

By Lemma 3.5 and inequality (4) we obtain that

$$\begin{aligned} &|t(F, v_{n,t}) - t(F, s_n)| \\ &\qquad \le 4\Big(\prod_{ij\in E(F)} d_{F_{i,j}}\Big) \sum_{ij\in E(F)} \|v_{F_{i,j},n,t} - (s_n)_{F_{i,j}}\|_\square \le \frac{1}{t}d_{F_{i,j}}. \end{aligned}$$

Let $n \to \infty$ ($n \in Q_t$), then the left hand side tends to $|t(F, u_t) - A_F|$ by (7) and (5). Letting $t \to \infty$, (6) implies that $A_F = t(F, w)$ as claimed. $\square$

Now we are ready to complete the proof of our main theorem.

Proof of Theorem 2.6. Let $Q \subseteq \mathbb{R}^{\mathcal{F}}$ denote the set of all $\mathcal{F}$-moment sequences. Clearly, Q is a compact, convex set in $\mathbb{R}^{\mathcal{F}}$.

Let $s_n \in \mathcal{S}$ be the $\mathcal{F}$-moment representation of W_n. Applying Lemma 3.7, we get that there is a $w \in \mathcal{S}$ such that $w(x, y)$ is an $\mathcal{F}$-moment sequence for all $x, y \in [0, 1]$, and $t(F, W_n) \to t(F, w)$ for all $\mathcal{F}$-decorated graph F. The $\mathcal{F}$-moment function sequence w defines a $\mathcal{K}$-graphon W by Lemma 3.6, which proves the Theorem. $\square$

References

[1] C. Borgs, J. Chayes, L. Lovász: Moments of Two-Variable Functions and the Uniqueness of Graph Limits, *Geom. Func. Anal.*

[2] C. Borgs, J. Chayes, L. Lovász, V.T. Sós, K. Vesztergombi: Counting graph homomorphisms, in: *Topics in Discrete Mathematics* (ed. M. Klazar, J. Kratochvil, M. Loebl, J. Matoušek, R. Thomas, P. Valtr), Springer (2006), 315–371.

[3] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, B. Szegedy, K. Vesztergombi: Graph Limits and Parameter Testing, Proc. 38th Annual ACM Symp. on Theory of Computing (2006), 261–270.

[4] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent Graph Sequences I: Subgraph frequencies, metric properties, and testing, *Advances in Math.* 219 (2008), 1801–1851.

[5] M. Freedman, L. Lovász, A. Schrijver: Reflection positivity, rank connectivity, and homomorphisms of graphs *J. Amer. Math. Soc.* **20** (2007), 37–51.

[6] A. Frieze, R. Kannan: Quick approximation to matrices and applications, *Combinatorica* **19**, 175–220.

[7] L. Lovász, B. Szegedy: Limits of dense graph sequences, *J. Comb. Theory B* **96** (2006), 933–957.

[8] L. Lovász, B. Szegedy: Szemerédi's Lemma for the analyst, *Geom. Func. Anal.* **17** (2007), 252–270.

[9] L. Lovász, B. Szegedy: The graph theoretic moment problem (manuscript)